Int. J. Solids Structures, 1977, Vol. 13, pp. 549–559. Pergamon Press. Printed in Great Britain

VIBRATION ANALYSIS OF VISCOELASTIC BODIES WITH SMALL LOSS TANGENTS†

ZVI HASHIN
Department of Solid Mechanics, Materials and Structures, School of Engineering, Tel-Aviv University, Tel-Aviv, Israel

(*Received* 15 *June* 1976; *revised* 26 *October* 1976)

Abstract—The correspondence principle for vibrations of viscoelastic bodies is specialized to the case of small loss tangents, resulting in considerable simplification: Analytical evaluation of oscillatory fields is greatly simplified; peak frequencies and peak amplitudes under forced vibrations can be simply and directly determined; numerical solution of viscoelastic vibration problems becomes no more complicated than that of elastic problems. Similar simplifications result for computation of real and imaginary parts of effective complex moduli of composite materials.

INTRODUCTION

The problem of the steady state vibrations of viscoelastic bodies can be conveniently treated in terms of a well-known correspondence principle. The principle states essentially that the solution to a viscoelastic vibration problem is obtained from the solution of a geometrically similar elastic vibration problem, by replacement of elastic moduli by corresponding complex moduli (see e.g. [1]).

The actual evaluation of viscoelastic solutions often requires cumbersome separation of complicated functions of complex moduli into real and imaginary parts. It is the purpose of the present work to establish a much simplified method of analysis which is valid for the case when the loss tangents of the viscoelastic material are small compared to unity. The method which will be developed is not only of great analytical simplicity but also considerably facilitates numerical analysis of viscoelastic vibration problems.

Practically speaking, loss tangents of viscoelastic materials rarely exceed the value 0.1. It appears that the method which will here be given is still very accurate for such loss tangent magnitude.

The idea of the small loss tangent approximation has been used before. The resulting simplifications for some particular cases have been discussed in [1]. Hashin[2–4] has utilized a small loss tangent approximation to evaluate convenient expressions for the real and imaginary parts of effective complex moduli of viscoelastic composites.

In the present work the small loss tangent approximation is developed in systematic fashion to evaluate internal fields in vibrating viscoelastic bodies as well as peak ("resonance") frequencies and amplitudes for the case of forced vibrations. Some of the results given here have been reported in [5]. Recently, Schapery[6] has also treated the forced vibration case based on small loss tangent approximation from a different point of view.

For the purpose of subsequent development, it is necessary to briefly recapitulate the classical vibrations correspondence principle. Let the viscoelastic body be bounded by the surface S. The viscoelastic properties of the body are characterized in the general anisotropic case by the complex moduli.

$$\tilde{C}_{ijkl}(\iota\omega) = C'_{ijkl}(\omega) + \iota C''_{ijkl}(\omega) \tag{1.1}$$

where latin subscripts range over 1, 2, 3, ω is the frequency, $\iota = \sqrt{-1}$, prime denotes real part and double prime denotes imaginary part.

Suppose the body is subjected to the boundary conditions

$$\begin{aligned} u_i(S,t) &= \tilde{u}_i^0(S)\,e^{\iota\omega t} \quad \text{on} \quad S_u \\ T_i(S,t) &= \tilde{T}_i^0(S)\,e^{\iota\omega t} \quad \text{on} \quad S_T \end{aligned} \qquad S_u + S_T = S. \tag{1.2}$$

†Supported by the Air Force Office of Scientific Research (AFOSR), under Grant 74-2591, through the European Office of Aerospace Research (EOAR), United States Air Force.

where u_i are displacements and T_i are tractions. The fields of displacement and stress in the body then have the forms

$$\begin{aligned} u_i(\underset{\sim}{x}, t) &= \tilde{u}_i(\underset{\sim}{x})\, e^{\iota\omega t} \\ \sigma_{ij}(\underset{\sim}{x}, t) &= \tilde{\sigma}_{ij}(\underset{\sim}{x})\, e^{\iota\omega t}. \end{aligned} \tag{1.3}$$

The stress-strain relation is

$$\begin{aligned} \tilde{\sigma}_{ij} &= \tilde{C}_{ijkl}\tilde{\epsilon}_{kl} \\ \tilde{\epsilon}_{kl} &= \tfrac{1}{2}(\tilde{u}_{k,l} + \tilde{u}_{l,k}). \end{aligned} \tag{1.4}$$

In the absence of body forces the space dependent parts $\tilde{u}_i$ of the displacements satisfy the differential equations

$$\tilde{C}_{ijkl}(\iota\omega)\tilde{u}_{k,lj} + \rho\omega^2\tilde{u}_i = 0 \tag{1.5}$$

where ρ is the density.

From (1.2) and (1.3) there follow the boundary conditions

$$\begin{aligned} \tilde{u}_i(S) &= \tilde{u}_i^\circ && \text{on} \quad S_u \\ \tilde{T}_i(S) &= \tilde{\sigma}_{ij}(S)n_j = \tilde{T}_i^\circ && \text{on} \quad S_T. \end{aligned} \tag{1.6}$$

If an elastic body of identical shape with moduli C_{ijkl} is subjected to (1.2) the elastic problem is mathematically identical, with complex moduli replaced by C_{ijkl}. Let the elastic solution be written

$$\begin{aligned} {}^e u_i(\underset{\sim}{x}, t) &= {}^e\tilde{u}_i(\underset{\sim}{x}, \underset{\sim}{C})\, e^{\iota\omega t} \\ {}^e\sigma_{ij}(\underset{\sim}{x}, t) &= {}^e\tilde{\sigma}_{ij}(\underset{\sim}{x}, \underset{\sim}{C})\, e^{\iota\omega t}. \end{aligned} \tag{1.7}$$

where $\underset{\sim}{C}$ is a compact notation for the elastic moduli C_{ijkl}. It follows that in the viscoelastic case

$$\begin{aligned} \tilde{u}_i(\underset{\sim}{x}) &= {}^e\tilde{u}_i(\underset{\sim}{x}, \tilde{\underset{\sim}{C}}) \\ \tilde{\sigma}_{ij}(\underset{\sim}{x}) &= {}^e\tilde{\sigma}_{ij}(\underset{\sim}{x}, \tilde{\underset{\sim}{C}}) \end{aligned} \tag{1.8}$$

where $\tilde{\underset{\sim}{C}}$ is a compact notation for $\tilde{C}_{ijkl}$.

Equations (1.8) concisely express the correspondence principle for vibrations of viscoelastic bodies: Once the elastic solution is known, it is merely necessary to replace in it elastic moduli by complex moduli to obtain the space dependent parts of viscoelastic solution.

A similar correspondence principle has been shown to apply[2, 3], for effective complex moduli of viscoelastic composites. Consider first a composite which consists of k purely elastic phases. Let the effective elastic moduli C^*_{ijkl} of the composite be written

$$C^*_{ijkl} = F_{ijkl}[\underset{\sim}{C}^{(1)}, \underset{\sim}{C}^{(2)}, \ldots \underset{\sim}{C}^{(k)}, \{g\}] \tag{1.9}$$

where $\underset{\sim}{C}^{(1)}, \underset{\sim}{C}^{(2)}, \ldots$ denote the elastic moduli of the phases and $\{g\}$ denotes the internal phase geometry. Now consider a composite of entirely identical phase geometry whose phases are viscoelastic. Then the effective complex moduli $\tilde{C}^*_{ijkl}$ are given by

$$\tilde{C}^*_{ijkl}(\iota\omega) = F_{ijkl}[\tilde{\underset{\sim}{C}}^{(1)}(\iota\omega), \tilde{\underset{\sim}{C}}^{(2)}(\iota\omega), \ldots, \tilde{\underset{\sim}{C}}^{(k)}(\iota\omega), \{g\}] \tag{1.10}$$

where $\tilde{\underset{\sim}{C}}^{(1)}, \tilde{\underset{\sim}{C}}^{(2)}, \ldots$ denote the phases complex moduli.

If the composite consists of elastic and viscoelastic phases then the elastic phase moduli are left unchanged in the replacement formula (1.10).

Numerous applications of (1.10) have been given in [3, 4].

2. THE CASE OF SMALL LOSS TANGENTS

Consider any complex modulus (1.1). The loss tangent is defined as

$$\tan \delta_{ijkl} = \frac{C''_{ijkl}}{C'_{ijkl}} \quad \text{(no sum on right side).} \tag{2.1}$$

For most viscoelastic materials which can still be classified as linear the loss tangent is a small number, which only rarely reaches the value 0.1. Consideration of smallness of loss tangent leads to considerable simplification which will now be explained.

Consider a function $f(x+\iota y)$ in the domain

$$0 \le y \le y_0 \qquad y_0 \ll x. \tag{2.2}$$

Let it be assumed that at points $(x, 0)$ the function is analytic and can therefore be expanded in a Taylor series. Within the domain (2.2) the function may be approximated by the first two terms of the Taylor series

$$f(x+\iota y) \cong f(x) + \iota y f'(x). \tag{2.3}$$

Similarly a function of several complex variables $f(x_1+\iota y_1, x_2+\iota y_2, \ldots, x_m+\iota y_m)$ in the domain

$$\begin{gathered} 0 \le y_1 \le y_0, \quad 0 \le y_2 \le y_0, \ldots 0 \le y_m \le y_0 \\ y_0 \ll x_1, x_2, \ldots x_m \end{gathered} \tag{2.4}$$

can be approximated by

$$f(x_1+\iota y_1, x_2+\iota y_2, \ldots x_m+\iota y_m) = f(x_1, x_2, \ldots x_m) + \iota\left(y_1\frac{\partial f}{\partial x_1} + y_2\frac{\partial f}{\partial x_2} + \cdots y_m\frac{\partial f}{\partial x_m}\right). \tag{2.5}$$

The relations obtained will first be applied to the viscoelastic solution (1.8). For simplicity consider an isotropic material with complex bulk and shear modulus given by

$$\begin{aligned} \tilde{K}(\iota\omega) &= K'(\omega) + \iota K''(\omega) \\ \tilde{G}(\iota\omega) &= G'(\omega) + \iota G''(\omega) \end{aligned} \tag{2.6}$$

and define the loss tangents by

$$\begin{aligned} \tan \delta_K &= \frac{K''}{K'} \ll 1 \\ \tan \delta_G &= \frac{G''}{G'} \ll 1. \end{aligned} \tag{2.7}$$

Interpreting (2.6) as complex variables $x_1+\iota y_1$, $x_2+\iota y_2$ in (2.5) it follows that (1.8) can be decomposed into real and imaginary parts in following fashion

$$\begin{aligned} \tilde{u}_i &= \tilde{u}'_i + \iota\tilde{u}''_i \\ \tilde{\sigma}_{ij} &= \tilde{\sigma}'_{ij} + \iota\tilde{\sigma}''_{ij} \end{aligned} \tag{2.8}$$

$$\begin{aligned} \tilde{u}'_i &\cong {}^e\tilde{u}_i[\underset{\sim}{x}, K'(\omega), G'(\omega)] \\ \tilde{\sigma}'_{ij} &= {}^e\tilde{\sigma}_{ij}(\underset{\sim}{x}, K'(\omega), G'(\omega)] \end{aligned} \tag{2.9}$$

$$\begin{aligned} \tilde{u}''_i &\cong K''(\omega)\frac{\partial \tilde{u}'_i}{\partial K'} + G''(\omega)\frac{\partial \tilde{u}'_i}{\partial G'} \\ \tilde{\sigma}''_{ij} &\cong K''(\omega)\frac{\partial \tilde{\sigma}'_{ij}}{\partial K'} + G''(\omega)\frac{\partial \tilde{\sigma}'_{ij}}{\partial G'}. \end{aligned} \tag{2.10}$$

It is seen that the real parts of the viscoelastic solution are merely the elastic solution in which elastic moduli are replaced by (frequency dependent) real parts of complex moduli. The imaginary part of the viscoelastic solution is expressed in terms of products of imaginary parts of the complex moduli multiplied by partial derivatives of real parts of the solution with respect to real parts of complex moduli.

In view of (2.7), (2.10) can also be written in the form

$$\tilde{u}''_i = \tan\delta_K K' \frac{\partial \tilde{u}'_i}{\partial K'} + \tan\delta_G G' \frac{\partial \tilde{u}'_i}{\partial G'}$$
$$\tilde{\sigma}''_{ij} = \tan\delta_K K' \frac{\partial \tilde{\sigma}'_{ij}}{\partial K'} + \tan\delta_G G' \frac{\partial \tilde{\sigma}'_{ij}}{\partial G'}. \tag{2.11}$$

The results (2.8)–(2.11) can be generalized in obvious fashion to cases where the solution depends on any number of complex moduli. Pertinent examples for dependence on more than two complex moduli are anisotropic and/or heterogeneous bodies.

Some interesting implications of the results obtained will now be discussed. It is well known that in general the frequency dependence of the imaginary part of a complex modulus is much stronger than that of the real part. It is consequently seen that the real parts (2.9) are weakly frequency dependent while that of the imaginary parts (2.10) is roughly expressed by a linear combination of $K''(\omega)$ and $G''(\omega)$. In particular, $K''(\omega)$ can be neglected with respect to $G''(\omega)$ for most isotropic materials. In that case there result the proportionality relations

$$\tilde{u}''_i \cong G''(\omega) \frac{\partial \tilde{u}'_i}{\partial G'}$$
$$\tilde{\sigma}''_{ij} \cong G''(\omega) \frac{\partial \tilde{\sigma}'_{ij}}{\partial G'}. \tag{2.12}$$

The most important implications of the results are probably for numerical analysis of viscoelastic vibrations. Suppose that a computer program for numerical analysis of an elastic vibration problem is available. Such a problem can be directly utilized to compute (2.9). Now to compute (2.10) the partial derivatives must be determined numerically. This can be done in terms of the approximation

$$\frac{\partial \tilde{u}'_i}{\partial G'} \cong \frac{1}{\Delta G'} [\tilde{u}'_i(\underline{x}, K', G' + \Delta G') - \tilde{u}'_i(\underline{x}, K', G')] \tag{2.13}$$

and similarly for the other partials. Thus it is merely necessary to evaluate differences between neighboring *elastic* solutions.

Special treatment is needed for the case of resonance. When an elastic system is subjected to forcing inputs at frequency equal to a critical frequency of the system, the displacement becomes infinite. If the system is viscoelastic there are no infinities. Instead the displacements assume finite peaks, which can be very large at certain peak frequencies.

To consider such cases let the forced vibration solution for the displacement of an elastic system be written in the form

$$^e u(\underline{x}, t) = \frac{N(\underline{x}, \omega, M)}{D(\omega, M)} e^{i\omega t} \tag{2.14}$$

where M denotes dependence on one or several elastic moduli. In general N and D are also functions of the dimensions of the elastic system.

The elastic resonance condition is given by

$$D(\omega, M) = 0. \tag{2.15}$$

The frequency solutions of (2.15) define the critical frequencies $^e\omega_n$.

If the same system is viscoelastic then by the correspondence principle

$$u(\underline{x}, t) = \tilde{u}(\underline{x})\, e^{\iota\omega t}$$
$$\tilde{u}(\underline{x}) = \frac{N(\underline{x}, \omega, \tilde{M})}{D(\omega, \tilde{M})} \tag{2.16}$$
$$\tilde{M} = M'(\omega) + \iota M''(\omega)$$

where $\tilde{M}$ is the complex modulus corresponding to M.

Assuming small loss tangents and analyticity of the functions N and D in the domain

$$0 \leq M'' \leq M_0 \qquad M_0 \ll M' \tag{2.17}$$

it follows by previous development that

$$\tilde{u} = \frac{N' + \iota N''}{D' + \iota D''} \tag{2.18}$$

where

$$N' \cong N[\underline{x}, \omega, M'(\omega)] \qquad \text{(a)}$$
$$D' \cong D[\omega, M'(\omega)] \qquad \text{(b)}$$
$$N'' \cong M'' \frac{\partial N'}{\partial M'} \qquad \text{(c)} \tag{2.19}$$
$$D'' \cong M'' \frac{\partial D'}{\partial M'} \qquad \text{(d).}$$

The absolute value $|\tilde{u}|$ of $\tilde{u}$ is given by

$$|\tilde{u}| = \sqrt{\left(\frac{N'^2 + N''^2}{D'^2 + D''^2}\right)} \tag{2.20}$$

and since the imaginary parts are small compared to the real parts it may be shown that (2.20) assumes a maximum when

$$D' = D[\omega, M'(\omega)] = 0. \tag{2.21}$$

Equation (2.21) defines the peak frequencies ω_n. Since M' is frequency dependent, the peak frequencies ω_n will be shifted with respect to elastic critical frequencies ω_n. The amount of shifting increases with the rate of change of M' as a function of ω.

At peak frequencies ω_n, (2.18) assumes the form

$$\tilde{u}_n = \frac{N''_n}{D''_n} - \iota \frac{N'_n}{D''_n} \tag{2.22}$$

where the subscripts n denote values at ω_n. Neglecting N''^2_n with respect to N'^2_n, (2.22) can be written as

$$\tilde{u}_n = \frac{N'_n}{D''_n} e^{-\iota\psi_n}$$
$$\tan\psi_n = \frac{N'_n}{N''_n}. \tag{2.23}$$

The time dependent peak displacement $\tilde{u}_n$ is then given by

$$\tilde{u}_n(\underline{x}, t) = \frac{N'_n}{D''_n} e^{\iota(\omega_n t - \psi_n)}. \tag{2.24}$$

It is seen that the peak amplitudes are given by

$$\text{Amp}\,\tilde{u}_n = \frac{N'_n}{D''_n} \tag{2.25}$$

and the phase shift of output with respect to input is given by $-\psi_n$ which for large enough ratio N'_n/N''_n can be approximated by $-(\pi/2)$.

The above given development defines a simple procedure for evaluation of peak frequencies and amplitudes for the case of small loss tangents. The pertinent equations are (2.21) and (2.25).

It should be noted that the development is the same for the case of dependence on several moduli. If in the elastic case there enter the moduli $M_1, M_2, \ldots M_k$ and the corresponding complex moduli are $\tilde{M}_1(\iota\omega), \tilde{M}_2(\iota\omega) \ldots \tilde{M}_k(\iota\omega)$, then (2.19) is replaced by

$$\begin{aligned} N' &\cong N[\underset{\sim}{x}, \omega, M'_1(\omega), M'_2(\omega), \ldots M'_k(\omega)] \\ D' &\cong D[\omega, M'_1(\omega), M'_2(\omega), \ldots M'_k(\omega)] \\ N'' &\cong \sum_{\ell=1}^{\ell=k} \frac{\partial N'}{\partial M'_\ell} M''_\ell \\ D'' &\cong \sum_{\ell=1}^{\ell=k} \frac{\partial D'}{\partial M'_\ell} M''_\ell \end{aligned} \tag{2.26}$$

and (2.21) is replaced by

$$D'[\omega, M'_1(\omega), M'_2(\omega), \ldots M'_k(\omega)] = 0. \tag{2.27}$$

Everything else remains the same.

The present procedure for determination of peak frequencies and amplitudes is not directly applicable to numerical analysis. Determination of peak frequencies requires numerical solution of (2.21) and determination of peak amplitudes requires numerical evaluation of (2.25). This, however, is not possible since by numerical analysis only the ratio N/D of (2.14) (the space dependent part of the elastic oscillating field) is known and not the functions N and D themselves.

This difficulty can be overcome in the following fashion: Define the reciprocal of the elastic amplitude by

$$\eta(\underset{\sim}{x}, \omega, M) = \frac{D(\omega, M)}{N(\underset{\sim}{x}, \omega, M)}. \tag{2.28}$$

In the viscoelastic case replace M by $M'(\omega)$. Thus

$$\eta'(\underset{\sim}{x}, \omega, M') = \frac{D[\omega, M'(\omega)]}{N[\underset{\sim}{x}, \omega, M'(\omega)]}. \tag{2.29}$$

The quantity η' can be found numerically by numerical solution of an elastic vibration problem in which $M'(\omega)$ is used instead of the elastic modulus. The frequency equation (2.21) is fulfilled when (2.29) vanishes. Thus the peak frequencies ω_n are determined by

$$\eta' = 0. \tag{2.30}$$

It should be noted that (2.30) vanishes for all $\underset{\sim}{x}$.

For the purpose of evaluation of (2.25), (2.29) is differentiated with respect to M'. Then

$$\frac{\partial \eta'}{\partial M'} = \frac{\dfrac{\partial D'}{\partial M'}}{N'} - \frac{\dfrac{\partial N'}{\partial M'} D'}{N'^2}. \tag{2.31}$$

The last term on the right side of (2.31) vanishes at ω_n in view of (2.29–30). Then by use of (2.19d), and (2.25) it follows that

$$\text{Amp } \tilde{u}_n = \frac{1}{\dfrac{\partial \eta'}{\partial M'} M''} \quad \text{at} \quad \omega = \omega_n. \tag{2.32}$$

To evaluate (2.32) numerically, write:

$$\left.\frac{\partial \eta'}{\partial M'}\right|_{\omega=\omega_n} \cong \frac{1}{2\Delta M'}[\eta'(\underline{x}, \omega_n, M'_n + \Delta M') - \eta'(\underline{x}, \omega_n, M'_n - \Delta M')] \tag{2.33}$$

where

$$M'_n = M'(\omega_n)$$

and $\Delta M'$ is an arbitrary small increment of M' at ω_n.

The small loss tangent approximation is readily applicable to viscoelastic composites. It is merely necessary to interpret (1.9) as the function f and the phase complex moduli as the complex variables $x_1 + \iota y_1$, $x_2 + \iota y_2, \ldots$ in (2.5).

Let $\tilde{M}^*$ be any effective complex modulus of a composite and ${}^eM^*$ the corresponding effective elastic modulus of the associated elastic composite. Write

$$M^* = M^{*\prime}(\omega) + \iota M^{*\prime\prime}(\omega). \tag{2.34}$$

Let M^* be dependent on the phase complex moduli $M_1, M_2, \ldots$ which are written

$$\begin{aligned} \tilde{M}_1 &= M'_1 + \iota M''_1 \\ \tilde{M}_2 &= M'_2 + \iota M''_2. \\ &\vdots \end{aligned} \tag{2.35}$$

Then

$$\begin{aligned} M^{*\prime} &= {}^eM^*(M'_1, M'_2, \ldots) & \text{(a)} \\ M^{*\prime\prime} &\cong M''_1 \frac{\partial M^{*\prime}}{\partial M'_1} + M''_2 \frac{\partial M^{*\prime}}{\partial M'_2} + \cdots & \text{(b)} \end{aligned} \tag{2.36}$$

or defining

$$\tan \delta_1 = \frac{M''_1}{M'_1} \ll 1, \quad \tan \delta_2 = \frac{M''_2}{M'_2} \ll 1 \ldots.$$

(2.36b) assumes the form

$$M^{*\prime\prime} \cong \tan \delta_1 M'_1 \frac{\partial M^{*\prime}}{\partial M'_1} + \tan \delta_2 M'_2 \frac{\partial M^{*\prime}}{\partial M'_2} + \cdots. \tag{2.37}$$

Results of type (2.36)–(2.37) have already been given and exploited in [2–4].

3. APPLICATION: TORSIONAL VIBRATIONS

As a simple example for the preceding theory consider torsional vibrations of a cylindrical rod which is built in at one edge and is subjected to a sinusoidal forcing torque at its other edge.

If the rod is elastic then the elementary one-dimensional version of the problem is given by

$$\begin{aligned} \frac{\partial^2 \phi}{\partial x^2} &= \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} \\ c^2 &= \frac{GI}{\rho J} \end{aligned} \tag{3.1}$$

$$\phi(o,t)=0 \qquad \text{built in edge}$$
$$M(l,t)=GJ\frac{\partial\phi}{\partial x}\bigg|_{x=l}=M_0\,e^{\iota\omega t} \qquad \text{loaded edge} \tag{3.2}$$

where

ϕ = angle of twist
ρ = density
I = polar moment of inertia
J = Saint Venant torsion coefficient.

The solution to this problem is given by

$$\phi(x,t)=\frac{M_0c}{\omega GJ}\frac{\sin(\omega x/c)}{\cos(\omega l/c)}e^{\iota\omega t} \tag{3.3}$$

The critical frequencies are defined by

$$\cos({}^e\omega_n l/c)=0$$
$${}^e\omega_n=\frac{(2n-1)\pi}{2}\frac{l}{c}. \tag{3.4}$$

If the rod is viscoelastic with complex modulus

$$\tilde{G}(\iota\omega)=G'(\omega)+\iota G''(\omega) \tag{3.5}$$

then by the correspondence principle

$$\phi(x,t)=\frac{M_0\tilde{c}}{\omega\tilde{G}J}\frac{\sin(\omega x/\tilde{c})}{\cos(\omega l/\tilde{c})}e^{\iota\omega t}$$
$$\tilde{c}^2=\frac{\tilde{G}I}{\rho J}. \tag{3.6}$$

Separation of (3.6) into real and imaginary parts is a tedious undertaking, resulting in complicated functions. These have then to be plotted as functions of ω in order to identify peak frequencies and peak amplitudes.

Considerable simplification is achieved by application of the theory of small loss tangents. For frequencies not close to peak frequencies, it follows from (1.3), (2.9–10) and (3.3) that

$$\phi(x,t)=(\phi'+\iota\phi'')e^{\iota\omega t}$$
$$\phi'=\frac{M_0c'}{\omega G''J}\frac{\sin(\omega x/c')}{\cos(\omega l/c')} \tag{3.7}$$

where

$$c'=\sqrt{\left(\frac{G'(\omega)}{\rho}\right)} \qquad \text{(a)}$$
$$\phi''=G''\frac{\partial\phi'}{\partial G'} \qquad \text{(b).} \tag{3.8}$$

For simplicity (3.8b) is evaluated at $x=l$ only. Thus

$$\phi''(l,t)=-\frac{M_0}{2\omega\rho Jc'}\left[\tan(\omega l/c')+\frac{\omega l/c'}{\cos^2(\omega l/c')}\right]\tan\delta\, e^{\iota\omega t}. \tag{3.9}$$

For evaluation of peak frequencies and peak amplitudes (2.14) is compared with (3.5). This provides the identification

$$\begin{aligned} N &= M_0 c \sin(\omega x/c) && \text{(a)} \\ D &= \omega G J \cos(\omega l/c) && \text{(b)} \\ M &= G && \text{(c)}. \end{aligned} \tag{3.10}$$

It follows from (2.21) and (3.10) that the peak frequencies ω_n are defined by

$$\cos(\omega l/c') = 0 \tag{3.11}$$

where c' is given by (3.8a). The solutions of (3.11) are given by

$$\omega_n^2 = \frac{(2n-1)\pi}{2} \frac{G'(\omega_n)}{\rho l}. \tag{3.12}$$

It is convenient to write $G'(\omega)$ in the form

$$G'(\omega) = G_0 f(\omega/\omega_0) \tag{3.13}$$

where ω_0 is some reference frequency and

$$G_0 = G'(\omega_0). \tag{3.14}$$

It is customary to represent f as a function of $\log(\omega/\omega_0)$ for convenience of plotting.

If the cylinder were elastic with shear modulus G_0 its resonant frequencies would be given by

$${}^e\omega_n^2 = \left(\frac{2n-1}{2}\pi\right)^2 \frac{G_0}{\rho l}. \tag{3.15}$$

Consequently, (3.12) can be written in the form

$$\omega_n^2 = {}^e\omega_n^2 f(\omega_n/\omega_0). \tag{3.16}$$

Equation (3.16) is easily solved by iteration, starting out with the value $\omega_n^{(1)} = {}^e\omega_n$.

Next, the amplitude peaks of the angle of twist ϕ are evaluated at the tip $x = l$. It follows from (2.19a, b) and (3.10) that

$$\begin{aligned} N_n' &= M_0 c' \sin(\omega_n l/c') && \text{(a)} \\ D_n' &= \omega_n G' J \cos(\omega_n l/c') && \text{(b)}. \end{aligned} \tag{3.17}$$

where c' is given by (3.8a). Now D_n'' is evaluated by use of (2.19d) and (3.17b). Then from (2.25) and (3.11) there results the simple expression

$$\text{Amp}\, \phi_n(l) = \frac{2M_0}{\rho J l \omega_n^2 \tan\delta_n} \tag{3.18}$$

where

$$\tan\delta_n = \frac{G''(\omega_n)}{G'(\omega_n)} \tag{3.19}$$

and ω_n are the peak frequencies defined by (3.12) or (3.16).

A similar calculation for ψ_n, the phase lag between input and peak output, as defined by (2.23), yields

$$\tan\psi_n = -\frac{2}{\tan\delta_n}. \tag{3.20}$$

In order to appreciate the simplicity of the results obtained, the form of the angle of twist obtained without the small loss tangent approximation will be given. It follows from (3.6) after lengthy calculation that

$$\text{Amp } \phi_n(l) = \frac{M_0 c}{\omega_n |G| J} \frac{\sqrt{(\sin^2 2\alpha + \sinh^2 2\beta)}}{\cos 2\alpha + \cosh 2\beta}$$

$$\psi_n = -\tan^{-1}\left(\frac{\sinh 2\beta}{\sin 2\alpha}\right) - \frac{\delta}{2}$$

where

$$|G| = G'(\omega_n)\sqrt{[1 + \tan^2 \delta)}$$

$$c = \sqrt{\left(\frac{|G|}{\rho}\right)}$$

$$\alpha = \frac{\omega_n l}{c} \cos \delta/2$$

$$\beta = \frac{\omega_n l}{c} \sin \delta/2$$

ω_n = frequency at maxima of (3.20).

To assess the accuracy of the small loss tangent approximation peak frequencies and peak amplitudes have been computed for a circular cylinder with the following data:

$l = 60''$ length
$d = 4.0''$ diameter
$\rho = 3.0$ density relative to water
$G'(\omega) = G_0(1 + \frac{1}{4}\log_{10}\omega)$ real part of shear modulus
$G_0 = 2 \times 10^6$ psi
$\tan \delta = 0.1$ frequency independent loss tangent†
$M_0 = 1000$ lb. in.

The frequency eqn (3.12) assumes the form

$$\omega_1^2 = \frac{\pi}{2} \frac{G_0}{\rho l}\left(1 + \frac{1}{4}\log_{10}\omega_1\right)$$
$$\omega_2^2 = \frac{3\pi}{2} \frac{G_0}{\rho l}\left(1 + \frac{1}{4}\log_{10}\omega_2\right) \quad \text{etc.} \tag{3.21}$$

Equations (3.21) are easily solved by iteration yielding the peak frequencies

$$\omega_1 = 3020.4 \text{ rad/sec.}$$

$$\omega_2 = 9353.7 \text{ rad/sec.}$$

The peak amplitudes as given by (3.18) and by (3.20) are compared below

	exact	small loss tangent approximation
ϕ_1	5.17226×10^{-3}	5.17396×10^{-3}
ϕ_2	0.54811×10^{-3}	0.53949×10^{-3}

†Strictly speaking, $\tan \delta$ must be also frequency dependent. The present assumption is merely for simplicity of calculation. All of previous developments are valid for frequency dependent loss tangent.

The phase lags at peak frequencies are close to $-90°$ for the present case of $\tan \delta = 0.1$. Usually, the loss tangent is smaller and consequently ψ_n is even closer to $-90°$.

4. CONCLUSION

A simplified method has been given to analyse forced vibrations of viscoelastic bodies and structures for the case of small loss tangents. Of particular importance is the simplified evaluation of peak frequencies and peak amplitudes.

The method can be conveniently used for numerical analysis of viscoelastic vibrating systems. In this case evaluation of real part of solution is simply a numerical elastic analysis in terms of real parts of complex moduli. Numerical evaluation of imaginary part of solution requires numerical evaluation of derivative of real part with respect to real part of complex modulus. In this respect it should be noted that numerical analysis of a vibrating viscoelastic system can also be directly carried out in terms of the classical version of the correspondence principle by use of a computer program with complex moduli. Choice of procedure to be adopted in numerical analysis of a specific problem will depend upon the circumstances.

A sample calculation has been carried out for the case of $\tan \delta = 0.1$ resulting in high accuracy for peak frequencies and amplitudes. Since loss tangents encountered in practice are usually smaller than 0.1, it is concluded that the method is very accurate for practical applications.

REFERENCES

1. R. M. Christensen, *Theory of Viscoelasticity*. Academic Press, New York (1971).
2. Z. Hashin, Complex moduli of viscoelastic composites—I. General theory and application of particulate composites. *Int. J. Solids Structures* **6**, 539 (1970).
3. Z. Hashin, Complex moduli of viscoelastic composites—II. Fiber reinforced materials. *Int. J. Solids Structures* **6**, 797 (1970).
4. Z. Hashin, *Theory of Fiber Reinforced Materials*. NASA CR 1974, (1972).
5. Z. Hashin, Vibrations of viscoelastic bodies with small loss tangents. *Sci. Report No.* 2. AFOSR Contract F44620-71-C-0100, Department of Materials Engineering, Technion, Haifa, Israel, July (1972).
6. R. A. Schapery, Viscoelastic behavior and analysis of composite materials. In *Mechanics of Composite Materials* (Edited by G. P. Sendeckyj). Academic, New York (1974).